

Extrinsically homogeneous hypersurfaces of the Thurston geometry Sol_0^4

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Thurston geometries are an interesting class of Riemannian homogeneous spaces, which play a fundamental role in geometrization theorems (i.e. the uniformization of compact orientable 2-manifolds and the Thurston geometrization for compact orientable 3-manifolds). Formally, a Thurston geometry is a pair (M, G) of a simply-connected manifold M and a Lie group $G \subset \text{Diff}(M)$ that acts transitively on M . Moreover, we require the stabilizers to be compact, which implies the existence of a G -invariant Riemannian metric on M , turning these manifolds into interesting objects for Riemannian geometers.

Due to their importance in 3-dimensional geometrization, the three-dimensional Thurston geometries (\mathbb{R}^3 , \mathbb{S}^3 , \mathbb{H}^3 , $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\text{SL}}_2\mathbb{R}$, Nil^3 and Sol^3) have been extensively studied, see for example [1] and [2] (parallel and totally geodesic submanifolds of space forms), [3], [4] and [5] (parallel and totally geodesic surfaces in the remaining 3-dimensional geometries). In dimension four, the space forms (\mathbb{R}^4 , \mathbb{S}^4 , \mathbb{H}^4 , $\mathbb{C}P^2$, $\mathbb{C}\mathbb{H}^2$) and products of lower-dimensional geometries also have been studied in detail, see for example [6] (parallel submanifolds of products of space forms). Only in recent years, the study of the four-dimensional geometries Sol_0^4 , $\text{Sol}_{m,n}^4$, \mathbb{F}^4 , Sol_1^4 and Nil^4 from a Riemannian point of view has started. We refer to among others [7] and [8].

In this talk we focus on 4-dimensional Thurston geometries and discuss a classification result of extrinsically homogeneous hypersurfaces in Sol_0^4 , that is, those hypersurfaces which are orbits of subgroups of $\text{Isom}_o(\text{Sol}_0^4)$. This classification is obtained by observing that certain functions related to a hypersurface in Sol_0^4 , are constant if the hypersurface is extrinsically homogeneous. Combining this information with the Gauss and Codazzi equations, we obtain several possibility for the immersion of the hypersurface. The last step consists of determining which of those immersions is in fact an orbit of a subgroup of $\text{Isom}_o(\text{Sol}_0^4)$.

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