Left invariant complex structures on nilpotent Lie groups

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Let G be a real simply connected nilpotent Lie group with the tangent Lie algebra $\mathfrak{g}$ and $J : \mathfrak{g} \to \mathfrak{g}$ define a left-invariant almost complex structure on G. Consider the complexification $\mathfrak{g}^*_{\mathbb{C}}$ of the dual space $\mathfrak{g}^*$ and its splitting

$$(\mathfrak{g}^*)^{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{1}$$

where $\Lambda^{1,0}$ and $\Lambda^{0,1}$ are eigen-spaces of $J^*_{\mathbb{C}}$ corresponding to $i, -i$, respectively. Salamon [1] proved that J is integrable if and only if there exists a basis $\omega_1, \dots, \omega_n$ of the space $\Lambda^{1,0}$ of $(1,0)$-forms such that

$$d\omega^{l+1} \in I(\omega^1, \dots \omega^l),\ l = 0, \dots, n-1, \tag{2}$$

where $I(\omega^1, \dots \omega^l)$ is the ideal generated by $\omega^1, \dots \omega^l$ and $d : \mathfrak{g}^*_{\mathbb{C}} \to \mathfrak{g}^*_{\mathbb{C}} \wedge \mathfrak{g}^*_{\mathbb{C}}$ is dual to the Lie bracket $[,]_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \wedge \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$. There are complex structures J (called nilpotent [2]) for which the stronger condition of decomposability of d is satisfied

$$d\omega^{l+1} \in \Lambda^2(\omega^1, \dots \omega^l, \bar{\omega}^1, \dots \bar{\omega}^l),\ l = 0, \dots, n-1. \tag{3}$$

In this case the differential algebra generated by $\omega^1, \dots \omega^l$ can be considered as a minimal model $\mathcal{M}^J_{\mathfrak{g}}$ of a nilmanifold G/Γ, where Γ is a cocompact lattice. The existence of nilpotent complex structure J imposes restrictions on the structure of $\mathfrak{g}$. We introduce the notion of a special minimal model $\mathcal{M}^J_{\mathfrak{g}}$ for an arbitrary nilpotent Lie algebra $\mathfrak{g}$ with an integrable complex structure of J. With its help we classify the following two classes of real nilpotent Lie algebras that admit an integrable complex structure [3]:

1) 8-dimensional nilpotent Lie algebras $\mathfrak{g}$ with $b_1(\mathfrak{g}) = 2$;

2) narrow naturally graded Lie algebras.

References

[1] S.M. Salamon, *Complex Structures on Nilpotent Lie Algebras*, J.Pure Appl.Algebra, **157** (2001), 311–333.

[2] L.A. Cordero, M. Fernandez, A. Gray, L. Ugarte, *Compact nilmanifolds with nilpotent complex structures: Dolbeault cohomology*, Trans. Amer. Math. Soc. **352** (2) (2000), 5405–5433.

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