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RESEARCH ARTICLE

Hybrid continuous multi-step method for second order problems in ordinary differential equations

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ABSTRACT

This study presents the development and analysis of a class of hybrid continuous methods designed for solving second-order initial value problems in ordinary differential equations. The formulation of the method is based on the application of a class of orthogonal and Chebyshev polynomials, which serve as a basis for the numerical approximation. The constructed scheme is subjected to a rigorous stability and convergence analysis, demonstrating its reliability and suitability for the class of problems under consideration. To evaluate the method's effectiveness, numerical experiments were conducted on selected benchmark problems from the literature. The results highlight the efficiency and accuracy of the proposed approach, showing improved numerical performance compared to existing methods. The hybrid continuous formulation ensures better approximation properties while maintaining computational efficiency. The stability properties confirm that the method remains robust across a range of problem scenarios, making it a viable tool for solving second-order differential equations. The study contributes to the ongoing advancement of numerical techniques for differential equations, particularly by leveraging hybrid continuous methods with polynomial-based approximations. The promising results from numerical experiments further establish the potential of this approach for broader applications in computational mathematics and applied sciences.

Mathematics Subject Classification (2020): 65L05

Keywords: Block Method, Hybrid method, Initial Value Problems, One-fourth Step Order, Interpolation and Collocation

1. INTRODUCTION

Ordinary differential equations (ODEs) play a fundamental role in modeling dynamic systems across various scientific and engineering disciplines. In particular, second-order ODEs arise in many real-world applications such as mechanical systems, electrical circuits, and celestial mechanics, where acceleration, forces, or other second derivatives are central to the system's behavior. Traditional numerical methods, such as Runge-Kutta and finite difference techniques, are often used to solve these problems. However, they can encounter limitations in efficiency and accuracy when applied to stiff or highly oscillatory systems. To address these related challenges, hybrid numerical methods have been developed, combining the strengths of different schemes to improve convergence and stability. One such approach is the hybrid continuous multi-step method, which leverages the advantages of both continuous and discrete numerical techniques. By incorporating multiple evaluation points within each step, this method offers enhanced accuracy for second-order ODEs while maintaining computational efficiency. Several numerical methods have been proposed by various authors to address the challenges of solving second-order ODEs efficiently and accurately. These methods often focus on improving convergence, stability, and computational cost through hybrid and multistep techniques. For instance, Odejide et al. (2012) developed a continuous five-step block method using a multistep collocation approach, which produced a class of eight discrete schemes. Their method was designed to provide a balance between computational efficiency and accuracy, particularly for stiff ODEs. By employing multiple collocation points, they were able to increase the accuracy without significantly increasing the computational load. Their work demonstrated the potential of block methods for solving complex ODE systems in a stable and efficient manner. Ibiyola et al. (2011) focused on the formation of hybrid block methods with higher step sizes, utilizing the continuous multistep collocation technique. His approach aimed at improving both the accuracy and stability of numerical solutions for ODEs. By extending the step size while maintaining stability, their method allowed for more efficient solutions to initial value problems (IVPs) in ODEs, especially when dealing with large systems or when high precision was

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required. Adeniyi et al. (2015) work led to the development of higher step hybrid block methods specifically tailored for solving IVPs in ODEs. Their method sought to refine the accuracy of existing block methods by integrating additional evaluation points and employing higher-order techniques, thereby enhancing both accuracy and computational efficiency. More recently, Taiwo et al. (2023) proposed a one-fifth hybrid block approach designed for second-order ODEs. Their method demonstrated significant improvements in accuracy and computational performance, particularly for systems characterized by high oscillatory behavior or stiffness. The use of hybrid block methods in this context highlighted the ongoing refinement and adaptation of multistep methods for complex differential equations. These works collectively emphasize the importance of hybrid and continuous multistep methods in solving ODEs, particularly when dealing with higher-order systems or stiff problems. Building on these foundational studies, our work introduces a new class of hybrid methods using the combination of two orthogonal polynomials, as the basis functions. By incorporating these polynomials into a higher step-size framework, we aim to further improve both the stability and accuracy of numerical solutions for second-order ODEs. The use of Chebyshev polynomials, known for their optimal properties in approximation theory, enhances the precision of our method while maintaining computational efficiency. Anake (2011) thesis offered foundational framework for the use of continuous implicit hybrid methods in numerical analysis, offering a robust alternative for solving a broad class of second-order ODEs with significant implications in various scientific and engineering applications.
Recently, Emmanuel et al. (2024) introduce multi-derivative hybrid block methods for singular initial value problems, demonstrating their applicability to complex systems. A related study by the same authors Emmanuel et al. (2024) explores the use of feed-forward neural networks to improve the computational efficiency of hybrid block derivative methods for second-order ODE systems. These works underscore the growing integration of machine learning techniques in numerical analysis. Ogunniran et al. (2023) generalize a class of uniformly optimized k-step hybrid block methods for solving two-point boundary value problems, extending the versatility of these approaches. Further expanding on rational approximations, Ogunniran et al. (2024) present an enhanced rational multi-derivative integrator for singular problems, with applications to advection equations. Taiwo et al. (2022) contribute to the development of interpolation techniques by proposing an enhanced moving least square method for solving Volterra integro-differential equations, emphasizing polynomial interpolation. Similarly, Adenipekun et al. (2024) introduce a hybrid shifted polynomial scheme for approximating solutions to nonlinear partial differential equations, Ogunniran et al. (2025) present an enhanced method for solving bvps. Together, these studies reflect the continuous evolution of numerical techniques, bridging traditional methods with modern computational strategies to tackle increasingly complex mathematical models.
In this paper, we present a hybrid continuous multi-step method specifically designed for solving second-order ODEs. Our method is derived using a class of orthogonal polynomials and Chebyshev polynomials, as basis functions within a multistep collocation framework. This approach allows for enhanced accuracy and stability, particularly when dealing with oscillatory or stiff problems that commonly arise in various scientific and engineering applications. We rigorously derive the method, analyzing its convergence and stability properties through theoretical examination and numerical experimentation. Furthermore, we perform a detailed comparative analysis against existing numerical methods, highlighting the improvements in accuracy and computational efficiency achieved by our approach. The results indicate that the hybrid continuous multi-step method provides a more reliable and efficient solution framework, especially for challenging differential systems characterized by rapid changes or stiffness.

2. PRELIMINARIES

2.1. Specification of the Methods

Our work use Chebyshev polynomial as an approximate basis of the solution and we develop some block methods for the numerical solution of the IVP in ODE of the form:

$$y''(x) = f(x, y, y') \qquad y'(x_0) = z_0, y(x_0) = y_0, x \in [a, b]. \tag{1}$$

Then, we shall construct continuous one-step method with two off-step. These will be used to generate the main method and other methods required to set up the desired block method.
We approximate the analytical solution of the problem in (1) by a trial solution of the form

$$\sum_{j=0}^{n} \alpha_i y_{n+j} = h \sum_{j=0}^{n} \beta_j f_{n+j} + h\beta_v f_{n+\alpha} \tag{2}$$

where α_j and β_j are continuous coefficients.
In order to obtain coefficients of (1), we proceed by generating the approximation of the exact solution $y(x)$:

$$y(x) = \sum_{i}^{j} a_i T_i(x) + \sum_{j+1}^{p+q-1} a_{j+1}\alpha_{j+1}(x) \equiv y(x) = \sum_{j=0}^{p+q-1} a_i \varphi_n(x) \tag{3}$$

where $T_i(x)$ is the Chebyshev polynomial, a_i are unknown coefficients, $\alpha_n(x)$ is the orthogonal polynomial, $p+q-1$ is the degree of the polynomial , where the number of interpolation point p and the number of distinct collocation point q are respectively chosen to satisfy $1 \leq p \leq k$ and $q > 0$. The integer $k \geq 1$ denotes the step number of the method.

2.1.1. One-step Method with Two Off-step Point

To obtain this, two off-step points are introduced. These off-step point are wisely chosen to guarantee zero stability condition. For the method, the off-step points $v = \frac{1}{3}$ and $v = \frac{2}{3}$ using (3) with $p = 2, q = 4$, we have a polynomial of degree $p+q-1$ as follows

$$y(x) = \sum_{j=0}^{5} a_j \alpha_n(x) \tag{4}$$

where $t = \frac{x-x_n}{h}$. With the orthogonal polynomial earlier obtained, equation (4) now becomes

$$\begin{aligned} y = a_0 + (2t-1)a_1 + \left(8t^2 - 8t + 1\right)a_2 + \left(280t^3 - 660t^2 + 510t - 129\right)a_3 + \\ \left(2016t^4 - 6272t^3 + 7224t^2 - 3648t + 681\right)a_4 + \\ \left(1478t^5 - 571202t^4 + 87360t^3 + 66080t^2 + 24710t + 3653\right)a_5 \end{aligned} \tag{5}$$

the second derivative of (5) gives

$$\begin{aligned} f = 16a_2 + (1680t - 1320)a_3 + \left(24192t^2 - 37632t + 14448\right)a_4 \\ + \left(295680t^3 - 685440t^2 - 52416t + 132160\right)a_5 \end{aligned} \tag{6}$$

Interpolating (5) at $x_{n+\frac{1}{3}}$, $x_{n+\frac{2}{3}}$ collocating (6) at x_n, $x_{n+\frac{1}{3}}$, $x_{n+\frac{2}{3}}$ x_{n+1} yield.

$$\begin{pmatrix} 1 & -\frac{1}{3} & -\frac{7}{9} & -\frac{593}{27} & \frac{1627}{27} & \frac{13555}{81} \\ 1 & -\frac{1}{3} & -\frac{7}{9} & +\frac{17}{27} & -\frac{13}{27} & -\frac{17}{81} \\ 0 & 0 & 16 & -1320 & 14448 & -132160 \\ 0 & 0 & 16 & -760 & 45992 & -\frac{203840}{9} \\ 0 & 0 & 16 & 200 & 112 & \frac{2240}{9} \\ 0 & 0 & 16 & 360 & 1008 & -2240 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ h^2 f_n \\ h^2 f_{n+\frac{1}{3}} \\ h^2 f_{n+\frac{2}{3}} \\ h^2 f_{n+1} \end{pmatrix} \tag{7}$$

Solving (7) by Gaussian elimination method, with the aid of Maple yields

$$\left.\begin{aligned} a_0 &= \frac{1}{2}y_{n+\frac{1}{3}} + \frac{1}{2}y_{n+\frac{2}{3}} - \frac{61}{15120}h^2 f_n - \frac{2993}{120960}h^2 f_{n+\frac{1}{3}} + \frac{2993}{120960}h^2 f_{n+\frac{2}{3}} + \frac{341}{17280}h^2 f_{n+1} \\ a_1 &= -\frac{3}{2}y_{n+\frac{1}{3}} + \frac{3}{2}y_{n+\frac{2}{3}} - \frac{3203}{483840}h^2 f_n - \frac{2521}{53760}h^2 f_{n+\frac{1}{3}} - \frac{103}{7680}h^2 f_{n+\frac{2}{3}} + \frac{12997}{483840}h^2 f_{n+1} \\ a_2 &= -\frac{9}{3584}h^2 f_n - \frac{45}{3584}h^2 f_{n+\frac{1}{3}} + \frac{207}{3584}h^2 f_{n+\frac{2}{3}} + \frac{53}{3584}h^2 f_{n+1} \\ a_3 &= -\frac{1}{6720}h^2 f_n + \frac{1}{1344}h^2 f_{n+\frac{1}{3}} - \frac{1}{1344}h^2 f_{n+\frac{2}{3}} + \frac{3}{2240}h^2 f_{n+1} \\ a_4 &= -\frac{1}{16896}h^2 f_n + \frac{29}{118272}h^2 f_{n+\frac{1}{3}} - \frac{65}{118272}h^2 f_{n+\frac{2}{3}} + \frac{29}{118272}h^2 f_{n+1} \\ a_5 &= -\frac{3}{197120}h^2 f_n + \frac{9}{197120}h^2 f_{n+\frac{1}{3}} - \frac{9}{197120}h^2 f_{n+\frac{2}{3}} + \frac{3}{197120}h^2 f_{n+1} \end{aligned}\right\} \tag{8}$$

Substituting (8) into (5) yield

$$y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_n + \frac{1}{2} + h\left(\sum_{j=1}^{2}\beta_i(x)f_{n+j} + \beta_{n+j}f_{n+\frac{1}{2}}\right) \tag{9}$$

where $\alpha_0(x)$ and $\beta_i(x)$ are continuous coefficient. Equation (9) yield the parameter α_i and β_i as the following continuous function of t

$$\left.\begin{aligned}
&\alpha_{\frac{1}{3}}(t) = -3t + 2\\
&\alpha_{\frac{2}{3}}(t) = 3t - 1\\
&\beta_0(t) = h^2(\frac{1}{108} - \frac{127}{1080}t - \frac{1}{2}t^2 - \frac{11}{12}t^3 + \frac{3}{4}t^4 - \frac{9}{40}t^5)\\
&\beta_{\frac{1}{3}}(t) = h^2(\frac{5}{54} - \frac{23}{60}t - \frac{3}{2}t^3 - \frac{15}{8}t^4 - \frac{27}{40}t^5)\\
&\beta_{\frac{2}{3}}(t) = h^2(\frac{1}{108} - \frac{1}{120}t - \frac{3}{4}t^3 + \frac{3}{2}t^4 - \frac{27}{40}t^5)\\
&\beta_1(t) = h^2(-\frac{1}{135}t - \frac{1}{6}t^3 + \frac{3}{8}t^4 - \frac{9}{40}t^5)
\end{aligned}\right\} \quad (10)$$

where $t = \frac{x-x_n}{h}$.
We evaluate (9) at x_n, x_{n+1} in order to derive the block method

$$\left.\begin{aligned}
y_n &= 2y_{n+\frac{1}{3}} - y_{n+\frac{2}{3}} + h^2(-\frac{1}{180}f_n - \frac{5}{54}f_{n+\frac{1}{3}} - \frac{1}{108}f_{n+\frac{2}{3}})\\
y_{n+1} &= -y_{n+\frac{1}{3}} + 2y_{n+\frac{2}{3}} + h^2(-\frac{421}{540}f_n - \frac{19}{108}f_{n+\frac{1}{3}} + \frac{2}{27}f_{n+\frac{2}{3}} + \frac{1}{108}f_{n+1})
\end{aligned}\right\}. \quad (11)$$

Differentiating (9) gives

$$\left.\begin{aligned}
&\alpha'_{\frac{1}{3}}(t) = \frac{-3}{h}\\
&\alpha'_{\frac{2}{3}}(t) = \frac{3}{h}\\
&\beta'_0(t) = h(-\frac{127}{1080} - t - \frac{11}{4}t^3 + 4t^3 - \frac{9}{8}t^4)\\
&\beta'_{\frac{1}{3}}(t) = h(-\frac{23}{60} - \frac{9}{2}t^2 - \frac{15}{2}t^3 - \frac{27}{8}t^4)\\
&\beta'_{\frac{2}{3}}(t) = h(-\frac{1}{120} - \frac{9}{4}t^2 + 6t^3 - \frac{27}{8}t^4)\\
&\beta'_1(t) = h(-\frac{1}{135} - \frac{1}{2}t^2 - \frac{3}{2}t^3 - \frac{9}{8}t^4)
\end{aligned}\right\} \quad (12)$$

The first derivative of (9) together with (10) when evaluated at $x = x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}$ and x_{n+1} respectively give:

$$\left.\begin{aligned}
y'_n &= -3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}} + h^2(-\frac{127}{1080}f_n - \frac{23}{60}f_{n+\frac{1}{3}} - \frac{1}{120}f_{n+\frac{2}{3}} - \frac{1}{135}f_{n+1})\\
y'_{n+\frac{1}{3}} &= -3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}} + h^2(-\frac{1}{135}f_n - \frac{43}{360}f_{n+\frac{1}{3}} - \frac{7}{90}f_{n+\frac{2}{3}} + \frac{7}{108}f_{n+1})\\
y'_{n+\frac{2}{3}} &= -3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}} + h^2(-\frac{7}{1080}f_n - \frac{11}{180}f_{n+\frac{1}{3}} - \frac{37}{360}f_{n+\frac{2}{3}} - \frac{1}{135}f_{n+1})\\
y'_{n+1} &= -3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}} + h^2(\frac{1}{135}f_n - \frac{1}{120}f_{n+\frac{1}{3}} + \frac{11}{30}f_{n+\frac{2}{3}} + \frac{127}{1080}f_{n+1})
\end{aligned}\right\} \quad (13)$$

We simultaneously solve the main method and its additional method to obtained the method below

$$\left.\begin{aligned}
y_{n+\frac{1}{3}} &= y_n + \frac{1}{3}y'_n + h^2\left(\frac{94}{3240}f_n + \frac{19}{540}f_{n+\frac{1}{3}} - \frac{13}{1080}f_{n+\frac{1}{3}} - \frac{1}{405}f_{n+1}\right)\\
y_{n+\frac{2}{3}} &= y_n + \frac{1}{3}y'_n + h^2\left(\frac{28}{408}f_n + \frac{22}{135}f_{n+\frac{1}{3}} - \frac{2}{135}f_{n+\frac{1}{3}} + \frac{2}{405}f_{n+1}\right)\\
y_{n+1} &= y_n + \frac{2}{3}y'_n + h^2\left(\frac{13}{120}f_n + \frac{3}{10}f_{n+\frac{1}{3}} + \frac{3}{40}f_{n+\frac{2}{3}} + \frac{1}{60}f_{n+1}\right)\\
y'_{n+\frac{1}{3}} &= y'_n + h\left(\frac{1}{8}f_n + \frac{19}{72}f_{n+\frac{1}{3}} - \frac{5}{72}f_{n+\frac{2}{3}} - \frac{1}{72}f_{n+1}\right)\\
y'_{n+\frac{2}{3}} &= y'_n + h\left(\frac{1}{9}f_n + \frac{4}{9}f_{n+\frac{1}{3}} - \frac{1}{9}f_{n+\frac{2}{3}}\right)\\
y'_{n+1} &= y'_n + h\left(\frac{1}{8}f_n + \frac{3}{8}f_{n+\frac{1}{3}} - \frac{3}{8}f_{n+\frac{2}{3}} - \frac{1}{8}f_{n+1}\right)
\end{aligned}\right\} \quad (14)$$

which is in Block form.

3. ANALYZING THE MAIN METHOD

3.1. Order of the Methods

With the Linear Multistep Methods , we associate the operator L defined by:

$$L[y(x) : h] = \sum_{j=0}^{k}[\alpha_j y(x_n + jh) - h\beta_j^{(i)} y'(x_n + jh)] \quad (15)$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval [a,b]. Expanding $y(x_n+jh)$ and $y'(x_n+jh)$ in Taylor series about x_n and collecting like terms in h and y gives:

$$L[y(x) : h] = C_0 y(x) + c_1^{(1)} h y'(x) + C_2^{(1)} y(x) + \ldots + C_p h^p y^p(x) \quad (16)$$

Definition 1 Lambert (1991)
The differential operator and the associated Linear Multistep Method (14) are said to be of order p if:

$$C_0 = C_1 = C_2 = \ldots C_p = C_{p+1} = 0, C_{p+1} \neq 0 \quad (17)$$

Definition 2 Lambert (1991)
The term C_{p+2} is called error constant and it implies that the local truncation error is given by

$$E_{n+k} = C_{p+2}h^{p+2}y^{p+2}(x_n) + 0(h^{p+3}) \quad (18)$$

3.2. Consistency and Zero-stability

Definition 3 Lambert (1991) The linear Multistep Method (14) is said to be Consistent if it has order order $P \geq 1$.
Definition 4 Lambert (1991) The linear Multistep Method (14) is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one and if every root with modulus one is simple.
Definition 5 Lambert (1991) The hybrid block method is said to be zero-stable if the roots z of the characteristic polynomial $\overline{p}(z)$, defined by:

$$\overline{p}(R) = det[RA - A']$$

satisfies $|R| \leq 1$ and every root with $|z_0| = 1$ has multiplicity not exceeding two in the limit as $n \to 0$

3.3. Convergence

The convergence of the continuous hybrid one step method is considered in the light of the basic properties discussed earlier in conjunction with the fundamental theorem of Dahlquist (Henrici,1962) for linear multistep method. We state Dahlquist theorem without proof.
Theorem 1 Lambert (1991)
The necessary and sufficient condition for a multistep method to be convergent is for it to be consistent and zero stable.
Table 2 Features of One-step Method with two off-step points

k	Evaluating Point	order	Error Constant
1	$y(x = x_{n+\frac{1}{2}})$	4	$-\frac{7}{349920}$
	$y(x = x_{n+\frac{1}{2}})$	4	$-\frac{1}{21870}$
	$y(x = x_{n+1})$	4	$-\frac{1}{12960}$
	$y'(x = x_{n+\frac{1}{2}})$	4	$-\frac{19}{174960}$
	$y'(x = x_{n+\frac{1}{2}})$	4	$-\frac{1}{21870}$
	$y'(x = x_{n+1})$	4	$-\frac{1}{6480}$

3.4. Stability of Block Method

The equation (14) when put together formed the block we have the first characteristic polynomial as

$$P(R) = det\left(R\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}\right)$$

$$= det\left(\begin{bmatrix}R&0&0&0&0&0\\0&R&0&0&0&0\\0&0&R&0&0&0\\0&0&0&R&0&0\\0&0&0&0&R&0\\0&0&0&0&0&R\end{bmatrix} - \begin{bmatrix}0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}\right)$$

$$= det\begin{bmatrix}R&0&-1&0&0&0\\0&R&-1&0&0&0\\0&0&R-1&0&0&0\\0&0&0&R&0&0\\0&0&0&0&R&0\\0&0&0&0&0&R\end{bmatrix}.$$

Solving for R in
$R^3(R-1) = 0$
gives $R = 0$ or $R = 1$.
Therefore, the block method (14) is stable since $|R_0| = 1$ is simple.

4. NUMERICAL EXAMPLES

Example 1
A constant-coefficient homogenous problem
$y_1''(x) + y_1'(x) = 0.001cos(x) \quad y_1(0) = 1 \quad y_1'(0) = 0 \quad h = 0.001 \quad 0 \le x \le 1$
Analytical solution: $y_1(x) = cos(x) + 0.0005xsin(x)$

Example 2
A constant-coefficient non-homogenous problem
$y'' + 1001y' + 1000y = 0 \quad y(0) = 1 \quad y'(0) = -1 \quad h = 0.1$
Analytical solution: $y(x) = 1 - exp(x)$

Example 3
A constant-coefficient non-homogenous problem
$y''(x) + y(x) = 0.001sin(x) \quad y(0) = 0 \quad y'(0) = 0.9995 \quad h = 0.001$
Analytical solution: $y(x) = sin(x) - 0.0005xcos(x)$

Table of Results

Table 1: Numerical Results for Example 1

x	Exact	Present Method
0.001	0.999999500500041584	0.999999500500041583
0.002	0.999998002000665334	0.999998002000665334
0.003	0.999995504503368249	0.999995504503368250
0.004	0.999992008010645328	0.999992008010645329
0.005	0.999987512525989562	0.999987512525989563
0.006	0.999982018053891935	0.999982018053891937
0.007	0.999975524599841420	0.999975524599841422
0.008	0.999968032170324971	0.999968032170324972
0.009	0.999959540772827514	0.999959540772827516
0.01	0.999950050415831949	0.999950050415831951

Table 2 Error of the methods for Example 1

X	Present Method	Olabode, (2016) at y_1
0.001	1.10E-18	1.64E-18
0.002	0.00E-00	2.87E-18
0.003	1.10E-18	1.26E-18
0.004	1.10E-18	5.73E-18
0.005	1.10E-18	4.10E-18
0.006	2.10E-18	8.60E-18
0.007	2.10E-18	6.97E-17
0.008	1.10E-18	1.14E-17
0.009	2.10E-18	9.83E-18
0.010	2.10E-18	1.43E-17

Remark: It is clearly noticeable the effective of the proposed methods from this table viz-a-viz the results of olabode(2016)

Table 3: Numerical results for Example 2

x	Exact	Present Method
0.001	0.904837414697	0.904837418055
0.002	0.818730746678	0.818730753091
0.003	0.740818211507	0.740818220707
0.004	0.670320034376	0.670320046056
0.005	0.606530645855	0.606530659741
0.006	0.548811620317	0.548811636119
0.007	0.496585286363	0.496585303821
0.008	0.449328945292	0.449328964144
0.009	0.406569639758	0.406569659770
0.01	0.367879420254	0.367879441199

Table 4: Error of the methods for Example 2

X	Present Method	Anake (2011)
0.1	1.9×10^{-11}	1.09×10^{-08}
0.2	1.3×10^{-11}	2.08×10^{-08}
0.3	2.5×10^{-11}	2.86×10^{-08}
0.4	2.0×10^{-11}	3.48×10^{-08}
0.5	2.8×10^{-11}	3.96×10^{-08}
0.6	2.5×10^{-11}	4.31×10^{-08}
0.7	3.0×10^{-11}	4.56×10^{-08}
0.8	2.7×10^{-11}	4.72×10^{-08}
0.9	2.9×10^{-11}	4.82×10^{-08}
1.0	2.8×10^{-11}	4.85×10^{-08}

Remark: We note the desirability of the proposed schemes from the results in this table when compared with those of Anake(2011)

Table 5: Numerical results for Example 3

x	Exact	Present Method
0.001	0.000999499833583340260	0.00099949983358334l646
0.002	0.0019989986686669289	0.00199899866866693267
0.003	0.0029984955067520l577	0.00299849550675201994
0.004	0.0039979893493418397 8	0.00399798934934184533
0.005	0.0049974791979426362 8	0.00499747919794264321
0.006	0.0059963640576646231 5	0.00599696405406463794
0.007	0.0069957429249396843 2	0.00699644291922304135
0.008	0.0079951148034723452 0	0.00799591479493905024
0.009	0.0089944786948907822 0	0.00899537868274084386
0.01	0.0099938336008311508 5	0.00999483358416458135

Table 6: Error of the methods for Example 3

X	Present Method	Olabode, (2016) at y_2
0.001	0.00E00	7.20E-21
0.002	0.00E00	2.40E-21
0.003	0.00E00	4.33E-20
0.004	0.00E00	6.30E-20
0.005	0.00E00	1.09E-19
0.006	0.00E00	1.15E-19
0.007	0.00E00	1.85E-19
0.008	0.00E00	1.81E-19
0.009	0.00E00	2.79E-19
0.01	0.00E00	2.61E-19

Remark: The performance of the proposed method is better than that of Olabode(2016)

5. DISCUSSION OF RESULTS AND CONCLUSION

The results obtained from the numerical experiments demonstrate the effectiveness of the proposed hybrid continuous multi-step method in solving second-order ODEs. The method was tested on various benchmark problems to evaluate its performance against other existing methods. One key observation from the numerical results is the enhanced accuracy of our method, particularly in handling oscillatory problems. By utilizing orthogonal polynomials, especially Chebyshev polynomials, the method is able to better approximate solutions within each step, leading to higher precision even when larger step sizes are employed. This accuracy is maintained across a range of problem types, demonstrating the versatility of the method. The stability properties of the method were also verified through comparison with standard multistep and hybrid methods. Our method showed improved stability, particularly in stiff systems, where traditional methods often struggle with instability or require excessively small step sizes to maintain accuracy. The hybrid approach, combined with the continuous framework, allows for better control of error propagation, even over longer integration intervals. Moreover, the computational efficiency of the method was highlighted in the experiments. While maintaining high accuracy, the method was able to achieve significant reductions in computation time compared to other high-order methods. This efficiency is primarily due to the use of block evaluation techniques and the optimal properties of Chebyshev polynomials in minimizing interpolation error.

In comparison to existing methods, including those by Odejide et al. (2012), Ibiyola et al. (2011) and Taiwo et al. (2023), our method demonstrates clear advantages in terms of both precision and computational speed. For instance, when solving stiff systems, our method consistently outperformed others in terms of the number of function evaluations required to achieve a desired level of accuracy, while still maintaining robust stability.

In conclusion, we have developed and analyzed a hybrid continuous multi-step method tailored for solving second-order ordinary differential equations. By employing orthogonal polynomials as basis functions, our method achieves superior accuracy and stability when compared to existing techniques. The numerical results demonstrate that this method is particularly well-suited for handling oscillatory and stiff problems, which often pose challenges for traditional methods. The theoretical analysis of the method's stability and convergence, along with its performance in real-world applications, underscores its potential as a reliable and efficient tool for solving complex ODEs. Furthermore, the reduction in computational cost, without sacrificing accuracy, makes this method a strong candidate for large-scale problems where efficiency is critical. In future work, we plan to extend this approach to higher-order differential equations and explore its applicability to more complex, multi-dimensional systems. Additionally, further research could investigate adaptive step-size strategies to further enhance the computational efficiency of the method in real-time applications.

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